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NORM INEQUALITIES INVOLVING ORDINARY AND JACOBI DERIVATIVES

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Abstract—This paper is concerned with norm inequalities for Jacobi weighted L^p spaces, namely with estimates of ordinary derivatives $\varphi^{2j} f^{(k)}$ and Jacobi derivatives $D_w^r f$ of Sturm-Liouville type in terms of $\|f\|_{w,p}$ and $\|\varphi^{2r} f^{(2r)}\|_{w,p}$, for suitable Jacobi weights w, φ.

1. INTRODUCTION

The Jacobi transform is a powerful tool in developing approximation theory in Jacobi weighted spaces, just as is the finite Fourier transform in the instance of periodic functions. By replacing the fundamental system $\{e^{ikx}\}_{k\in\mathbb{Z}}$ in the latter by Jacobi polynomials, one can define convolution integrals, summation processes, etc., to obtain weighted approximation theorems by algebraic polynomials on the bounded interval $[-1,1]$ (cf. [1–5]). For instance, the well-known Bernstein-Durrmeyer polynomials are thus covered in this frame [6]. The disadvantage of this approach is that the ordinary translation $T_t f(x) = f(x+t)$ and derivatives have to be replaced by the Jacobi translation and by Jacobi derivatives of Sturm-Liouville type. This leads to calculational difficulties in the applications. Thus, our central aim is to obtain an estimate of the r^{th} Jacobi $D_w^r f$ derivative by ordinary derivatives.

Let $L_w^p \equiv L_w^p(-1,1)$, $1 \le p < \infty$, denote the Banach space of real-valued Lebesgue measurable functions f on $(-1,1)$, endowed with the norm

$$\|f\|_{w,p} \equiv \|f\|_{L_w^p} := \left\{\int_{-1}^{1} |f(x)|^p w(x)\,dx\right\}^{1/p},$$

$w(x) \equiv w_{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$, $\alpha,\beta > -1$, being the Jacobi weight. For $w \equiv 1$, we simply write $L^p(-1,1)$. Considering derivatives in L_w^p, the natural weight $\varphi(x) := \sqrt{1-x^2}$ occurs. To this end, we define the weighted Sobolev space $W_{w,p}^r$, $r \in \mathbb{N}$, by

$$W_{w,p}^r := \left\{f \in L_w^p;\ f^{(r-1)} \in AC_{\text{loc}}(-1,1),\ \varphi^r f^{(r)} \in L_w^p\right\},$$

where $AC_{\text{loc}}(-1,1)$ is the set of the locally absolutely continuous functions on $(-1,1)$. Let m, M denote positive constants, not necessarily the same at each occasion, which do not depend on the elements of L_w^p. Finally, we write $[x]$ for the integer part of a real number x.

To solve our problem, we first generalize some well-known estimates involving ordinary derivatives of different orders (Theorem 1, Corollary 1). In Theorem 2, we give a representation for the Jacobi derivative $D_w^r f$ of r^{th} order in terms of ordinary derivatives, whereas the first Jacobi derivative can be simply written as (see Lemma 3)

$$D_w^1 f(x) = -\frac{1}{2(\alpha+1)}\,\frac{1}{w(x)}\,\frac{d}{dx}\left\{w(x)\varphi^2(x)\,\frac{d}{dx}f(x)\right\}, \qquad x \in (-1,1).$$

The final step then is to apply the estimates of Theorem 1 to the representation of Theorem 2.

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2. WEIGHTED ESTIMATES OF ORDINARY DERIVATIVES

Concerning classical derivatives our first theorem now reads

THEOREM 1. *For $f \in W^{2r}_{w,p}$, $r \in \mathbb{N}$, $1 \le p < \infty$, there holds the estimate*

$$\left\|\varphi^{2j} f^{(k)}\right\|_{w,p} \le M\left\{\|f\|_{w,p} + \left\|\varphi^{2r} f^{(2r)}\right\|_{w,p}\right\}, \tag{1}$$

for all $j, k \in \mathbb{N}_0$ with $0 \le k \le 2r$, $j \ge k - r$.

The nonweighted case, i.e., $w(x) \equiv 1$, was established by Ditzian and Totik [7, p. 135]. For $j \ge k/2$, Theorem 1 is covered by Theorem 3.5 in [8]. However, results of this type go back among others to work done by G. H. Hardy, J. E. Littlewood and E. Landau (1932/35), I. Halperin and H. Pitt (1938), and L. Nirenberg (1955); see [9, Chapter I] and the literature cited there.

We split the proof of Theorem 1 into several lemmas. On compact subintervals of $(-1,1)$, the weight w behaves like the identity, in particular,

$$w \sim 1 \quad \text{on } \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{2}$$

i.e., there are constants $m, M > 0$ such that $m \le w(x) \le M$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Therefore, we can use classical nonweighted estimates on compact subintervals of $(-1,1)$. Concerning the endpoints of the interval, observe that

$$w(x) \sim (1-x)^{\alpha} \quad \text{on } \left[-\frac{1}{2}, 1\right), \quad w(x) \sim (1+x)^{\beta} \quad \text{on } \left(-1, \frac{1}{2}\right]. \tag{3}$$

In the following, we use (2), (3) for the weights w and $\varphi = w_{(1/2,1/2)}$ frequently without explicit references. The basic tool to solve the estimate (1) at the endpoints is the Hardy inequality, stating that (cf., e.g., [10, p. 245])

$$\left\{\int_0^\infty t^{\gamma-1}\left(\int_t^\infty |g(u)|\,du\right)^p dt\right\}^{1/p} \le \frac{p}{\gamma}\left\{\int_0^\infty |ug(u)|^p u^{\gamma-1}\,du\right\}^{1/p}, \qquad \gamma > 0,\ 1 \le p < \infty, \tag{4}$$

for every Lebesgue measurable function g on $(0,\infty)$.

LEMMA 1. *Let $f \in W^{2r}_{w,p}$, $r \in \mathbb{N}$, and suppose that* $\operatorname{supp}(f) \subset (-1, \frac{1}{2}]$ *or* $\operatorname{supp}(f) \subset [-\frac{1}{2}, 1)$, *then*

$$\left\|\varphi^{2(r-k)} f^{(2r-k)}\right\|_{w,p} \le M\left\|\varphi^{2r} f^{(2r)}\right\|_{w,p}, \qquad 0 \le k \le r. \tag{5}$$

PROOF. By symmetry, we may restrict our attention to the case $\operatorname{supp}(f) \subset \left(-1, \frac{1}{2}\right]$, and show that for $s, j \in \mathbb{N}_0$, $s+1 \le 2r$, $j+1 \le r$,

$$\left\|\varphi^{2j} f^{(s)}\right\|_{w,p} \le M\left\|\varphi^{2(j+1)} f^{(s+1)}\right\|_{w,p}.$$

Then (5) follows by simple induction. Indeed, let f be extended from $(-1,1)$ to $(-1,\infty)$ by zero, then

$$\int_x^\infty \left|f^{(s+1)}(u)\right| du \ge \left|\int_x^1 f^{(s+1)}(u)\,du\right| = |f^{(s)}(x)|, \qquad x \in (-1,1).$$

Applying the Hardy inequality (4) with $\gamma = jp + \beta + 1 > 0$, we obtain

$$\begin{aligned}
\left\|\varphi^{2j} f^{(s)}\right\|_{w,p} &\le M\left\|(1+x)^{j+\beta/p}\int_x^\infty \left|f^{(s+1)}(u)\right| du\right\|_{L^p(-1,\infty)} \\
&= M\left\{\int_0^\infty \left(\int_t^\infty \left|f^{(s+1)}(v-1)\right| dv\right)^p t^{jp+\beta}\,dt\right\}^{1/p} \\
&\le M\frac{p}{jp+\beta+1}\left\{\int_0^\infty \left|vf^{(s+1)}(v-1)\right|^p v^{jp+\beta}\,dv\right\}^{1/p} \\
&\le M\left\|\varphi^{2(j+1)} f^{(s+1)}\right\|_{w,p}.
\end{aligned}$$

∎

Now we combine the estimates at the endpoints and an estimate inside of $(-1,1)$. This will be carried out in our next lemma, the proof being adapted from the nonweighted case in [7, p. 135].

LEMMA 2. *For* $f \in W^{2r}_{w,p}$, $r \in \mathbb{N}$, *and* $r \le k \le 2r$, *there hold*

$$\left\|\varphi^{2(k-r)} f^{(k)}\right\|_{w,p} \le M\left\{\|f\|_{w,p} + \left\|\varphi^{2r} f^{(2r)}\right\|_{w,p}\right\}.$$

PROOF. On $\left[-\frac{1}{2},\frac{1}{2}\right]$, we use the classical estimate for functions $g, g^{(2r)} \in L^p(a,b)$, $[a,b] \subset \mathbb{R}$, given by (see, e.g., [8, Theorem 1.2] or [11])

$$\left\|g^{(j)}\right\|_{L^p(a,b)} \le M\left\{\|g\|_{L^p(a,b)} + \left\|g^{(2r)}\right\|_{L^p(a,b)}\right\}, \qquad 0 \le j \le 2r.$$

This implies that for $0 \le j \le 2r$, we have

$$\left\|f^{(j)}\right\|_{L^p(-1/2,1/2)} \le M\left\{\|f\|_{w,p} + \left\|\varphi^{2r} f^{(2r)}\right\|_{w,p}\right\}. \tag{6}$$

Choose $\theta \in C^\infty[-1,1]$, satisfying $\theta(x) = 1$, $x \le -\frac{1}{2}$, $\theta(x) = 0$, $x \ge \frac{1}{2}$, and define $f_1 := \theta f$, $f_2 := (1-\theta)f$. We now estimate the $(2r)^{\text{th}}$ derivative of f_i, $i = 1,2$, by

$$\left\|\varphi^{2r} f_i^{(2r)}\right\|_{w,p} \le M\left\{\|f\|_{w,p} + \left\|\varphi^{2r} f^{(2r)}\right\|_{w,p}\right\}. \tag{7}$$

To verify (7), we use Leibniz's rule for $f_1 = f - (1-\theta)f$ to obtain the representation

$$f_1^{(2r)} = f^{(2r)} - \sum_{\nu=0}^{2r} \binom{2r}{\nu} (1-\theta)^{(\nu)} f^{(2r-\nu)}.$$

Observing that $\operatorname{supp}(f_1) \subset \left(-1,\frac{1}{2}\right]$, $\operatorname{supp}(1-\theta) \subset \left[-\frac{1}{2},1\right)$ we find, noting $\varphi \le 1$ and (6),

$$\begin{aligned}
\left\|\varphi^{2r} f_1^{(2r)}\right\|_{w,p} &= \left\|\varphi^{2r} f_1^{(2r)} w^{1/p}\right\|_{L^p(-1,1/2)} \\
&\le \left\|\varphi^{2r} f^{(2r)} w^{1/p}\right\|_{L^p(-1,1/2)} + M\sum_{\nu=0}^{2r} \left\|\varphi^{2r}(1-\theta)^{(\nu)} f^{(2r-\nu)} w^{1/p}\right\|_{L^p(-1,1/2)} \\
&\le \left\|\varphi^{2r} f^{(2r)}\right\|_{w,p} + M\sum_{\nu=0}^{2r} \left\|f^{(2r-\nu)}\right\|_{L^p(-1/2,1/2)} \\
&\le M\left\{\|f\|_{w,p} + \left\|\varphi^{2r} f^{(2r)}\right\|_{w,p}\right\}.
\end{aligned}$$

The estimate for f_2 follows along the same lines. We complete the proof by combining Lemma 1 with (6) and (7), to yield

$$\begin{aligned}
\left\|\varphi^{2(k-r)} f^{(k)}\right\|_{p,w} &\le \left\|\varphi^{2(k-r)} f^{(k)} w^{1/p}\right\|_{L^p(-1,-1/2)} + \left\|\varphi^{2(k-r)} f^{(k)} w^{1/p}\right\|_{L^p(-1/2,1/2)} \\
&\quad + \left\|\varphi^{2(k-r)} f^{(k)} w^{1/p}\right\|_{L^p(1/2,1)} \\
&\le \left\|\varphi^{2(k-r)} f_1^{(k)}\right\|_{w,p} + M\left\|f^{(k)}\right\|_{L^p(-1/2,1/2)} + \left\|\varphi^{2(k-r)} f_2^{(k)}\right\|_{w,p} \\
&\le M\left\{\|f\|_{w,p} + \left\|\varphi^{2r} f^{(2r)}\right\|_{w,p}\right\}.
\end{aligned}$$

∎

In order to prove Theorem 1, we now use Lemma 2 iteratively to control the powers of φ.

PROOF OF THEOREM 1. Suppose first $r \le k \le 2r$, $j \ge k - r$, then Lemma 2 implies the estimate (1) by the boundedness of φ.

Now let $0 \le k < r$, $j \in \mathbb{N}_0$. For $k = 0$, there is nothing to prove, otherwise, using the boundedness of φ again, it suffices to estimate $f^{(k)}$. This will be done by choosing $m \in \mathbb{N}$ with $2^{m-1}k < r \le 2^m k$. A repeated application of Lemma 2 for $2^i k$ instead of r then yields

$$\begin{aligned}\left\|f^{(k)}\right\|_{w,p} &\le M\left\{\left\|f\right\|_{w,p} + \left\|\varphi^{2k} f^{(2k)}\right\|_{w,p}\right\}\\ &\le M\left\{\left\|f\right\|_{w,p} + \left\|f^{(2k)}\right\|_{w,p}\right\}\\ &\le \cdots \le M\left\{\left\|f\right\|_{w,p} + \left\|\varphi^{2^m k} f^{(2^m k)}\right\|_{w,p}\right\}\\ &\le M\left\{\left\|f\right\|_{w,p} + \left\|\varphi^{2r} f^{(2r)}\right\|_{w,p}\right\}.\end{aligned}$$

This completes the proof of Theorem 1. ∎

It becomes obvious that Theorem 1 can be extended to arbitrary positive weights having the behaviour (2) inside, and (3) at the endpoints of $(-1, 1)$.

Slight modifications of Lemmas 1 and 2 yield an analogous result for odd order derivatives.

COROLLARY 1. *Let* $f \in W^{2r+1}_{w,p}$, $r \in \mathbb{N}_0$, $1 \le p < \infty$, *then*

$$\left\|\varphi^{2j+1} f^{(k)}\right\|_{w,p} \le M\left\{\left\|f\right\|_{w,p} + \left\|\varphi^{2r+1} f^{(2r+1)}\right\|_{w,p}\right\},$$

for all $k, j \in \mathbb{N}_0$ *with* $0 \le k \le 2r+1$, $j \ge k - r$.

3. ESTIMATES OF HIGHER ORDER JACOBI DERIVATIVES

In order to make the term "Jacobi derivative" precise, we need some basic facts from transform theory (see the literature cited above). For $f \in L^p_w$ the Jacobi transform is defined by

$$f^\frown(k) := \int_{-1}^{1} f(x) R_k(x) w(x)\, dx, \qquad k \in \mathbb{N}_0,$$

$R_k(x) \equiv R_k^{(\alpha,\beta)}(x) := P_k^{(\alpha,\beta)}(x)/P_k^{(\alpha,\beta)}(1)$ denoting the normalized Jacobi polynomial of degree $k \in \mathbb{N}_0$ (see, e.g., [12, Chapter IV]). The Jacobi transform is a bounded linear operator from L^p_w into c_0, the set of all sequences $a = \{a_k\}_{k\in\mathbb{N}_0}$ with $\lim_{k\to\infty} a_k = 0$, also satisfying the uniqueness theorem

$$f^\frown(k) = 0, \qquad \forall k \in \mathbb{N}_0 \text{ if and only if } f = 0. \tag{8}$$

Terms like $f = 0$ have to be understood in the sense of $\|f\|_{w,p} = 0$. The counterpart of the classical translation $T_t f(x) = f(x+t)$ is the Jacobi translation operator τ_t which may defined in view of the uniqueness theorem in terms of the Jacobi transform by

$$[\tau_t f]^\frown(k) := R_k(t) f^\frown(k), \qquad f \in L^p_w, \quad k \in \mathbb{N}_0, \quad t \in (-1, 1). \tag{9}$$

It was shown by Gasper [13,14] (see also [15]), that $\tau_t \colon L^p_w \to L^p_w$ is a bounded linear operator if and only if $\alpha \ge \beta > -1$, $\alpha + \beta \ge -1$, to which we restrict ourselves for the remaining section. We recall that

$$\lim_{t\to 1^-} \left\|f - \tau_t f\right\|_{w,p} = 0, \qquad f \in L^p_w.$$

In this frame, the (strong) Jacobi derivative $D^r_w f$, $r \in \mathbb{N}$, is iteratively defined by

$$D^1_w f := \operatorname*{s\text{-}lim}_{t\to 1^-} \frac{f - \tau_t f}{1 - t}, \quad D^r_w f := D^1_w(D^{r-1}_w f),$$

with corresponding Sobolev space $J\text{-}W^r_{w,p} := \{f \in L^p_w;\ D^r_w f \in L^p_w\}$.

First, we have to determine the Jacobi derivative in terms of ordinary derivatives.

LEMMA 3. *Let* $f \in J\text{-}W^1_{w,p}$, $1 \le p < \infty$, *then* $D^1_w f$ *has the (pointwise) representation*

$$\begin{aligned} D^1_w f(x) &= -\frac{1}{2(\alpha+1)}\,\frac{1}{w(x)}\,\frac{d}{dx}\left\{w(x)\,\varphi^2(x)\,\frac{d}{dx}\,f(x)\right\} \\ &= -\frac{1}{2(\alpha+1)}\left\{\varphi^2(x)f^{(2)}(x)+(\alpha+\beta+2)(c-x)f'(x)\right\}, \qquad x\in(-1,1), \end{aligned} \tag{10}$$

where $c \equiv c_{(\alpha,\beta)} := (\beta-\alpha)/(\alpha+\beta+2)$.

PROOF. Using (9), we obtain

$$[D^1_w f]^\frown(k) = R'_k(1)f^\frown(k) = \frac{k(k+\alpha+\beta+2)}{2(\alpha+1)}\,f^\frown(k), \qquad k\in\mathbb{N}_0,$$

with $\varphi^2 w f'$ belonging to $AC_{\mathrm{loc}}(-1,1)$, and vanishing in -1 and 1, (cf. [1, Theorem 7.1.1.; 6]). On the other hand, R_k satisfy the Sturm-Liouville differential equation $\varphi^2(x)Y''(x)+(\alpha+\beta+2)(c-x)Y'(x)+k(k+\alpha+\beta+2)Y(x)=0$. Denoting the right-hand side of (10) by $\tilde{D}_w f$, integration by parts implies

$$[\tilde{D}_w f]^\frown(k) = \frac{k(k+\alpha+\beta+2)}{2(\alpha+1)}f^\frown(k), \qquad k\in\mathbb{N}_0.$$

Finally, the uniqueness theorem (8) proves the lemma. ∎

An useful explicit representation of the iterated derivative $D^r_w f$ becomes too complicated. Instead, we state a qualitative result.

THEOREM 2. *Let* $f \in J\text{-}W^r_{w,p}$, $1\le p<\infty$, $r\in\mathbb{N}$, *then there hold*

$$D^r_w f = \sum_{j=1}^{2r} p_{r,j} f^{(j)}, \tag{11}$$

where $p_{r,j}$ *are polynomials*

$$p_{r,j}(x) = \sum_{i=j_r}^{[j/2]} \left(\gamma_{r,j,i}+\delta_{r,j,i}(c-x)\right)\varphi^{2i}(x), \qquad x\in(-1,1), \tag{12}$$

with $j_r = \max\{j-r,0\}$. *The real constants* $\gamma_{r,j,i}$, $\delta_{r,j,i}$ *are independent of* f *and* x, *satisfying* $\gamma_{r,1,0}=0$ *and* $\delta_{r,j,j/2}=0$ *for even* j.

PROOF. We give an outline of the proof, because a detailed elaboration would take several pages of elementary calculations. The case $r=1$ is given by Lemma 3. By induction on r, we have to show that the term

$$D^{r+1}_w f = \sum_{j=1}^{2r} D^1_w(p_{r,j}f^{(j)}) \tag{13}$$

has the representation (11) with (12) for $r+1$. Therefore, we have to discuss functions of the type $F(x) := (\gamma+\delta(c-x))\varphi^{2i}f^{(j)}(x)$, $1\le j\le r$, $j_r\le i\le [j/2]$. Calculations show that

$$\begin{aligned} D^1_w F(x) = -\frac{1}{2(\alpha+1)}\Bigg\{&\Big(4i(i-1)\gamma+2i(c\delta+\gamma)(1-c^2)(\alpha+\beta+2) \\ &+2i\big((2i-1)\delta+(c\gamma+\delta+c^2\delta)(\alpha+\beta+2)\big)(c-x)\Big)\varphi^{2i-2}(x) \\ &+\Big(2i\big(2c\delta+(1-2i)\gamma-(c\delta+\gamma)(\alpha+\beta+2)\big) \end{aligned}$$

$$
\begin{aligned}
&- (1+2i)\delta(2i+\alpha+\beta+2)(c-x)\Big)\varphi^{2i}(x)\Big\} f^{(j)}(x) \\
&- \frac{1}{2(\alpha+1)}\Big\{\Big(4i(\delta - c^2\delta - c\gamma) + \delta(1-c^2)(\alpha+\beta+2) \\
&\quad + \big(4i(\gamma + c\delta) + (2c\delta+\gamma)(\alpha+\beta+2)\big)(c-x)\Big)\varphi^{2i}(x) \\
&\quad - \delta\big(4i+\alpha+\beta+4\big)\varphi^{2i+2}(x)\Big\} f^{(j+1)}(x) \\
&- \frac{1}{2(\alpha+1)}\Big\{\big(\gamma + \delta(c-x)\big)\varphi^{2i+2}(x)\Big\} f^{(j+2)}(x) \\
&=: F_{j,i-1}(x) + F_{j,i}(x) + F_{j+1,i}(x) + F_{j+1,i+1}(x) + F_{j+2,i+1}(x),
\end{aligned}
$$

with $F_{n,m}(x) := (\gamma_{n,m} + \delta_{n,m}(c-x))\,\varphi^{2m} f^{(n)}$ and constants $\gamma_{n,m}, \delta_{n,m}$ depending on γ, δ and i. Now we assume that F equals one of the terms of $p_{r,j} f^{(j)}$ in (13). An examination of the constants above implies that $F_{j,-1} \equiv 0$, $\gamma_{1,0} = 0$, $\delta_{n,n/2} = 0$ for even n, and finally that $\gamma_{n,m} = \delta_{n,m} = 0$ unless $1 \le n \le 2(r+1)$, $n_{r+1} \le m \le \left[\frac{n}{2}\right]$. ∎

It is worth mentioning that in the Gegenbauer case, i.e., $\alpha = \beta \ge -\frac{1}{2}$, the polynomials (12) reduce simply to

$$
p_{r,j}(x) = \sum_{i=j_r}^{j/2} \gamma_{r,j,i}\,\varphi^{2i}(x), \quad p_{r,j}(x) = \sum_{i=j_r}^{(j-1)/2} \delta_{r,j,i}\, x\,\varphi^{2i}(x), \qquad x \in (-1,1),
$$

for j even and j odd, respectively. This follows by similar arguments, noting that $c_{(\alpha,\alpha)} = 0$.

Applying Theorem 1, we immediately obtain our final result.

THEOREM 3. *Let $r \in \mathbb{N}$, $1 \le p < \infty$, $\alpha \ge \beta > -1$, $\alpha + \beta \ge -1$, then there hold $J\text{-}W^r_{w,p} \subset W^{2r}_{w,p}$ and the r^{th} Jacobi derivative can be estimated by*

$$
\left\|D^r_w f\right\|_{w,p} \le M\left\{\left\|f\right\|_{w,p} + \left\|\varphi^{2r} f^{(2r)}\right\|_{w,p}\right\}, \qquad f \in W^{2r}_{w,p}. \tag{14}
$$

We note that (14) holds for arbitrary $\alpha, \beta > -1$, if we would define the r^{th} Jacobi derivative $D^r_w f$ by iteration of the representation (10) in Lemma 3, instead of using the translation operator τ_t. Theorem 3 is a generalization of Lemma 3 in [6] with $r = 1$ to arbitrary $r \in \mathbb{N}$. A possible converse of (14) remains an open problem.

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